Polynomials 2

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HomeWork

Q) Show that (without multiplying it out),

$$\frac{b-c}{a} + \frac{c-a}{b} + \frac{a-b}{c} = \frac{(a-b)(b-c)(a-c)}{abc}$$

Ans:- $f(x) = \frac{b-c}{x} + \frac{c-x}{b} + \frac{x-b}{c}$

$f(b) = \frac{b-c}{b} + \frac{c-b}{b} + \frac{b-b}{c} = \frac{b-c}{b} - \frac{b-c}{b} + 0 = 0$

$f(c) = \frac{b-c}{c} + \frac{c-c}{b} + \frac{c-b}{c} = 0$

$f(x) = \frac{P(x)}{xbc}$ → Polynomial → because denominator has x, b, c in $f(x)$ and numerator will be a polynomial

$\Rightarrow P(b) = P(c) = 0$

$P(x) = (x-b)(x-c)Q(x)$ → Q is another polynomial

$$f(a) = \frac{(a-b)(a-c)Q(a)}{abc} = \frac{b-c}{a} + \frac{c-a}{b} + \frac{a-b}{c} \quad -①$$

$g(x) = \frac{x-c}{a} + \frac{c-a}{x} + \frac{a-x}{c}$

$g(a) = g(c) = 0$

$$g(b) = \frac{(b-a)(b-c)R(b)}{abc} = \frac{b-c}{a} + \frac{c-a}{b} + \frac{a-b}{c} \quad -②$$

We know order of P is ≤ 3

So Q & R is of degree 0 or 1

So for Q & R of degree 0 case is invalid

So Q & R of degree 1.

$\Rightarrow f(a) = g(b)$ — by ① & ②

$$\Rightarrow \frac{(a-b)(a-c)Q(a)}{\cdot} = \frac{(b-a)(b-c)R(b)}{\cdot} \quad -④$$

$$\Rightarrow \frac{(a-b)(a-c)\,Q(a)}{abc} = \frac{(b-a)(b-c)\,R(b)}{abc} \quad -④$$

$$\Rightarrow (a-c)\,Q(a) = (c-b)\,R(b) \quad -③$$

From ③ & ④ & ① & ② , we get,

$$\frac{b-c}{a} + \frac{c-a}{b} + \frac{a-b}{c} = k\left(\frac{(a-b)(b-c)(c-a)}{abc}\right)$$

$\rightarrow k$ is a constant

Put $a=-1,\ b=1,\ c=2$

$$\frac{-1}{-1} + \frac{3}{1} + \frac{-2}{2} = k\left(\frac{(-2)(-1)(3)}{(-1)(1)(2)}\right)$$

$$\Rightarrow 1+3-1 = k(-3)$$

$$\Rightarrow k = -1$$

$$\Rightarrow \frac{b-c}{a} + \frac{-a}{b} + \frac{a-b}{c} = \frac{(a-b)(b-c)(a-c)}{abc}$$

<u>Definition:-</u> Polynomials over set A is defined as polynomials of the form $a_0 + a_1 x + \cdots + a_n x^n$ where $a_i \in A$, $n \in \mathbb{N} + \{0\}$

A is a set.

$A[x]$ is the set of polynomials over A.

$$A[x] = \{a_0 + a_1 x + \cdots + a_n x^n : a_i \in A,\ n \in \mathbb{N} + \{0\}\}$$

If A is closed under addition and multiplication then

If A is closed under addition and multiplication then we get that $A[x]$ also closed under addition and multiplication.

$\mathbb{R}, \mathbb{Z}, \mathbb{Q}, \mathbb{N}$ are such sets.

$$P_1(x) = a_0 + a_1 x + \cdots + a_n x^n = \sum a_i x^i$$

$$P_2(x) = b_0 + b_1 x + \cdots + b_m x^m = \sum b_i x^i$$

$$P_3(x) = P_1(x) P_2(x) = c_0 + c_1 x + \cdots + c_{n} x^{m+n} = \sum c_i x^i$$

$$\Rightarrow \quad c_i = a_0 b_i + a_1 b_{i-1} + \cdots + a_i b_0 = \sum_{\substack{h+g=i \\ h, g \geq 0}} a_h b_g$$

$\hookrightarrow$ coefficient of x^i in P_3.

(*) Let $f(x)$ and $g(x)$ be polynomials in $A[x]$ where A is one of $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ or $\mathbb{Z}_n$. Then

$$f(x) = Q(x) g(x) + R(x)$$

where $Q(x), R(x) \in A[x]$ and $R(x)$ has degree less than that of $g(x)$. $Q(x)$ is the quotient and $R(x)$ is the remainder.

<u>Example</u> :- $f(x) = x^3 + x^2 + 7$
$g(x) = x^2 + 3 \longrightarrow \mathbb{Z}[x]$

$$f(x) = (x+1)\, g(x) + (-3x + 4) \to \in \mathbb{Z}[x]$$

$\in \mathbb{Z}[x]$

$$\begin{array}{r} x + 1 \\ x^2+3 \,\big)\, \overline{x^3 + x^2 + 7} \\ -\; x^3 + 3x \\ \hline x^2 - 3x + 7 \end{array}$$

$$\begin{array}{r} -\underline{x^3 \pm 3x} \\ x^2-3x+7 \\ -\underline{x^2 \pm 3} \\ 3x+4 \end{array}$$

Q) What is the largest integer value x can take such that x^3+100 is divisible by $x+10$.

Ans:— $x^3+100 = (x+10)(x^2-10x+100) - 900$

$\Rightarrow x+10 \mid -900$

$\Rightarrow x+10 = 900$ for largest value $\Rightarrow x = 890$

(*) Remainder Theorem:—

If a polynomial $P(x)$ is divided by $x-a$ then the remainder will be $P(a)$

$P(x) = (x-a)Q(x) + R(x)$

$\Rightarrow P(a) = 0 + R(a)$

as $x-a$ is of order 1

$R(a) = P(a)$ must be constant

so $R(a) = R(x)$ so $P(a)$ is remainder

(*) Factor Theorem:—

If a is a zero of polynomial $P(x)$, then, $x-a$ must be a factor of $P(x)$, i.e,

$P(x) = (x-a)Q(x)$ where $Q(x)$ is another polynomial.

⊛ Fundamental Theorem of Algebra :-

Every polynomial in $\mathbb{C}[x]$ has at least one complex zero.

⊛ Any n-th degree polynomial has exactly n complex zeros, although some of them may not be distinct.

Q> If $P(x)$ denotes polynomial of degree n such that $P(k) = \frac{k}{k+1}$ for $k = 0, 1, 2, \ldots, n$ determine $P(n+1)$.

Ans :- HomeWork. Hint :- Use the zeros of the polynomial

Q> Prove that the polynomial

$x^{2n} - 2x^{2n-1} - 3x^{2n-2} - \cdots - 2nx + 2n + 1$ has no real roots.

Ans :- HomeWork